

Quantum coherences, K -way negativities and multipartite entanglement

S. Shelly Sharma*

Depto. de Física, Universidade Estadual de Londrina, Londrina 86051-990, PR Brazil

N. K. Sharma†

Depto. de Matemática, Universidade Estadual de Londrina, Londrina 86051-990 PR, Brazil

A characterization of multipartite quantum states having N subsystems, based on negativities of matrices obtained by selective partial transposition of state operator, is proposed. The K -way partial transpose with respect to a subsystem is constructed by imposing constraints involving the states of K subsystems of multipartite composite system. The K -way negativity, defined as the negativity of K -way partial transpose, quantifies the K -way coherences of the composite system. For an N -partite system the fraction of K -way negativity ($2 \leq K \leq N$), contributing to global negativity, is obtained. The entanglement measures for a given state $\hat{\rho}$ are identified as the partial K -way negativities of the corresponding canonical state.

PACS numbers: 03.67.Mn, 03.65.Ud, 03.67.-a

Positive partial transpose (PPT), first introduced by Peres [1], is the most widely used separability criterion for quantum states. It has been shown to be a necessary and sufficient condition [2] for the separability of qubit-qubit and qubit-qutrit systems. For higher dimensional systems, positive partial transpose is a necessary condition [3, 4]. Negativity [5, 6] based on Peres Horodecki PPT criterion has been shown to be an entanglement monotone [6, 7]. In a multipartite quantum system composed of N subsystems, a single subsystem may be entangled to $(N-1)$ systems in distinctly different ways. For example, in a three qubit system (ABC) , the subsystem A can have genuine tripartite entanglement, W -like entanglement, as well as bipartite entanglement with subsystem B or C alone. The negativity of partial transpose of state operator $\hat{\rho}^{ABC}$ with respect to A , may, thus have distinct contributions that can be related to genuine tripartite or bipartite entanglement. The bipartite entanglement, may in turn be for the pair AB , the pair AC , or both the pairs. In a recent article [8], we have discussed the entanglement of three qubit states using 2-way and 3-way negativities. In this article, a characterization of multipartite quantum states having N subsystems, based on negativities of matrices obtained by selective partial transposition of state operator, is proposed. The K -way partial transpose with respect to a subsystem is constructed by imposing constraints involving the states of K subsystems of multipartite composite system. The K -way negativity ($2 \leq K \leq N$), defined as the negativity of K -way partial transpose, quantifies the K -way coherences of the composite system. The underlying idea of selective transposition to construct a K -way partial transpose with respect to a subsystem, first presented in ref. [9], shifts the focus from K -subsystems to K -way coherences of the composite system. By K -way coherences, we mean the quantum correlations responsible for GHZ state like entanglement of a K -partite system. For an N -partite entangled state, the negativity of global partial transpose is found to contain contributions from K -way partial transposes ($2 \leq K \leq N$). Entanglement is invariant under local unitary rotations, whereas, coherences are not so. The elements in the set of states obtained by performing entanglement conserving local operations on $\hat{\rho}$ differ from each other by the number of local basis product states and the number of variables required to write the state. In addition, the states in the set differ from each other by the K -way negativities characterizing the states. A pure state $\hat{\rho} = |\Psi\rangle\langle\Psi|$ may be mapped by local entanglement conserving operations to an operator $\hat{\rho}_c = |\Psi\rangle_{cc}\langle\Psi|$ such that the canonical state $|\Psi\rangle_c$ is a linear combination of minimum number of local basis product states (LBPS) [10]. The entanglement measures for a given state $\hat{\rho}$ are identified as the partial K -way negativities of the corresponding canonical state $\hat{\rho}_c$. A simple method to construct a state canonical to a given three qubit pure state has been given by Acin et al [11]. While GHZ like N -partite entanglement of a composite system is generated by N -way coherences, N -partite entanglement in general can be present due to K -way ($2 \leq K < N$) coherences as well. On the other hand, a system having K -partite entanglement may not have K -way coherences at all, as is the case of states having bound entanglement.

In section I, we define the global negativity, the K -way partial transpose of an N -partite state and the K -way negativity. Decomposition of K -way negativity into contributions from different components of a subsystem is discussed in section II. Section III deals with the contributions of K -way negativities to the global negativity. In section IV, the effect of local operations on coherences is discussed. A brief discussion on the connection between number of eigenvalues of N -way partial transpose and the canonical form, in the context of N -qubit state, is given in

*Electronic address: shelly@uel.br

†Electronic address: nsharma@uel.br

section V. Analytical expressions for entanglement measures of a special set of four parameter two qubit one qutrit state in Schmidt form are given in section VI, to illustrate the usage.

I. THE GLOBAL AND K-WAY NEGATIVITY OF N-PARTITE SYSTEM

The Hilbert space, $C^d = C^{d_1} \otimes C^{d_2} \otimes \dots \otimes C^{d_N}$, associated with a quantum system composed of N sub-systems, is spanned by basis vectors of the form $|i_1 i_2 \dots i_N\rangle$, where $i_m = 0$ to $(d_m - 1)$, d_m being the dimension of Hilbert space associated with m^{th} sub-system. The state operator for a general N-partite state is

$$\hat{\rho} = \sum_{\substack{i_1 i_2 \dots i_N, \\ j_1 j_2 \dots j_N}} \langle i_1 i_2 \dots i_N | \hat{\rho} | j_1 j_2 \dots j_N \rangle |i_1 i_2 \dots i_N\rangle \langle j_1 j_2 \dots j_N|. \quad (1)$$

The global partial transpose of $\hat{\rho}_G^{T_p}$ with respect to sub-system p is obtained from the matrix ρ by imposing the condition

$$\langle i_1 i_2 \dots i_N | \hat{\rho}_G^{T_p} | j_1 j_2 \dots j_N \rangle = \langle i_1 i_2 \dots i_{p-1}, j_p, i_{p+1}, \dots i_N | \hat{\rho} | j_1 j_2 \dots j_{p-1}, i_p, j_{p+1}, \dots j_N \rangle. \quad (2)$$

The partial transpose $\hat{\rho}_G^{T_p}$ of a state having free entanglement is non positive. Global Negativity, defined as

$$N_G^p = \frac{1}{d_p - 1} \left(\left\| \rho_G^{T_p} \right\|_1 - 1 \right), \quad (3)$$

measures the entanglement of subsystem p with its complement in a bipartite split of the composite system. Here $\|\rho\|_1$ is the trace norm of ρ . Global negativity vanishes on PPT-states and is equal to the entropy of entanglement on maximally entangled states.

A given matrix element $\langle i_1 i_2 \dots i_N | \hat{\rho} | j_1 j_2 \dots j_N \rangle$ is characterized by a number $\sum_{m=1}^N (1 - \delta_{i_m, j_m}) = K$, where $\delta_{i_m, j_m} = 1$ for $i_m = j_m$, and $\delta_{i_m, j_m} = 0$ for $i_m \neq j_m$. In other words, the total number of subsystems in bra vector in a state different from that in the ket vector in a matrix element $\langle i_1 i_2 \dots i_N | \hat{\rho} | j_1 j_2 \dots j_N \rangle$ is equal to K . The K -way partial transpose ($2 \leq K \leq N$) of N-partite state $\hat{\rho}$ with respect to subsystem p is obtained from matrix ρ by applying the following constraints:

$$\begin{aligned} \langle i_1 i_2 \dots i_N | \hat{\rho}_K^{T_p} | j_1 j_2 \dots j_N \rangle &= \langle i_1 i_2 \dots i_{p-1}, j_p, i_{p+1}, \dots i_N | \hat{\rho} | j_1 j_2 \dots j_{p-1}, i_p, j_{p+1}, \dots j_N \rangle, \\ \text{if } \sum_{m=1}^N (1 - \delta_{i_m, j_m}) &= K, \quad \text{and} \\ \langle i_1 i_2 \dots i_N | \hat{\rho}_K^{T_p} | j_1 j_2 \dots j_N \rangle &= \langle i_1 i_2 \dots i_N | \hat{\rho} | j_1 j_2 \dots j_N \rangle \\ \text{if } \sum_{m=1}^N (1 - \delta_{i_m, j_m}) &\neq K. \end{aligned} \quad (4)$$

The K -way negativity [8, 9] calculated from K -way partial transpose of matrix ρ with respect to subsystem p , is defined as

$$N_K^p = \frac{1}{d_p - 1} \left(\left\| \rho_K^{T_p} \right\|_1 - 1 \right). \quad (5)$$

Using the definition of trace norm and $\text{tr}(\rho_K^{T_p}) = 1$, we get

$$N_K^p = \frac{2}{d_p - 1} \sum_i |\lambda_i^{K-}|, \quad (6)$$

λ_i^{K-} being the negative eigenvalues of matrix $\rho_K^{T_p}$. The negativity N_K^p depends on K -way coherences and is a measure of all possible types of entanglement attributed to K -way coherences. Intuitively, for a system to have pure N -partite entanglement, it is necessary that N -way coherences are non-zero. On the other hand, N -partite entanglement can be generated by $(N - 1)$ -way coherences, as well. For a three qubit system, maximally entangled tripartite GHZ state is an example of genuine tripartite entanglement involving 3-way coherences. For maximally

entangled three qubit GHZ state the global negativity $N_G^p = N_3^p = 1$. Maximally entangled W-state is a manifestation of tripartite entanglement due to 2-way coherences having $N_G^p = N_2^p = 0.94$ and $N_3^p = 0$. For pure states the free entanglement of a subsystem is completely determined by the global Negativity and the hierarchy of negativities N_K^p ($K = 2, \dots, N$), calculated from $\rho_K^{T_p}$ associated with the p^{th} sub-system. For N partite system, $N_2^p, N_3^p, \dots, N_N^p$ and N_G^p ($p = 1$ to N) quantify the coherences characterizing bipartite splits of the system having p^{th} subsystem as one part.

II. HOW DO DIFFERENT SUBSYSTEMS CONTRIBUTE TO THE GLOBAL PARTIAL TRANSPOSE?

The negativity of partial transpose of a two qubit state

$$\hat{\rho}^{AB} = \sum_{i_1 i_2 j_1 j_2} \langle i_1 i_2 | \hat{\rho}^{AB} | j_1 j_2 \rangle | i_1 i_2 \rangle \langle j_1 j_2 | \quad (7)$$

depends on $\Delta = \langle 11 | \hat{\rho}^{AB} | 00 \rangle - \langle 01 | \hat{\rho}^{AB} | 10 \rangle$ and $\Delta^* = \langle 00 | \hat{\rho}^{AB} | 11 \rangle - \langle 10 | \hat{\rho}^{AB} | 01 \rangle$. The position of two pairs of matrix elements with $K = 2$ is exchanged to construct the partial transpose. For N subsystems, the set of K distinguishable subsystems that change state while $N-K$ of the sub-systems do not, can be chosen in $D_K = \frac{N!}{(N-K)!K!}$ distinct ways. However, the number of matrix elements that are transposed to get $\rho_K^{T_p}$ from ρ depends on D_K and the dimensions d_1, d_2, \dots, d_N of the subsystems. For a three qubit system (ABC), for example, the two independent contributions to $\rho_2^{T_A}(ABC)$ involve the qubit pairs AB and AC , respectively. Consider the three qubit pure state

$$\Psi^{ABC} = \sqrt{\mu_0} |000\rangle + \sqrt{\mu_1} \left(\frac{|110\rangle + |101\rangle + |111\rangle}{\sqrt{3}} \right), \quad \hat{\rho}^{ABC} = |\Psi^{ABC}\rangle \langle \Psi^{ABC}|, \quad (8)$$

with $\mu_0 + \mu_1 = 1$, $\sqrt{\mu_i} \geq 0$. It is common practice to trace out subsystem A to obtain the entanglement of B and C . State reduction is an irreversible local operation and it is believed that the entanglement of the pair BC in the reduced system is either the same or less than that in the composite system $\hat{\rho}^{ABC}$. One can, however, obtain a measure of 2-way coherences involving a given pair of subsystems from 2-way partial transpose constructed by restricting the transposed matrix elements of $\hat{\rho}^{ABC}$ to those for which the state of the third subsystem does not change. For example, $\rho_2^{T_{A-AB}}$ is obtained from the matrix ρ^{ABC} by applying the condition

$$\begin{aligned} \langle i_1 i_2 i_3 | \hat{\rho}_2^{T_{A-AB}} | j_1 j_2 i_3 \rangle &= \langle j_1 i_2 i_3 | \hat{\rho} | i_1 j_2 i_3 \rangle; \\ \text{if } \sum_{m=1}^3 (1 - \delta_{i_m, j_m}) &= 2, \\ \langle i_1 i_2 i_3 | \hat{\rho}_2^{T_{A-AB}} | j_1 j_2 j_3 \rangle &= \langle i_1 i_2 i_3 | \hat{\rho} | j_1 j_2 j_3 \rangle; \quad \text{for all other matrix elements.} \end{aligned} \quad (9)$$

Similarly, matrix elements of $\hat{\rho}_2^{T_{A-AC}}$ are related to matrix elements of the state operator by

$$\begin{aligned} \langle i_1 i_2 i_3 | \hat{\rho}_2^{T_{A-AC}} | j_1 i_2 j_3 \rangle &= \langle j_1 i_2 i_3 | \hat{\rho} | i_1 i_2 j_3 \rangle; \\ \text{if } \sum_{m=1}^3 (1 - \delta_{i_m, j_m}) &= 2, \\ \langle i_1 i_2 i_3 | \hat{\rho}_2^{T_{A-AC}} | j_1 j_2 j_3 \rangle &= \langle i_1 i_2 i_3 | \hat{\rho} | j_1 j_2 j_3 \rangle; \quad \text{for all other matrix elements.} \end{aligned} \quad (10)$$

The negativities $N_2^{A-AB} = \left(\left\| \hat{\rho}_2^{T_{A-AB}} \right\|_1 - 1 \right)$ and $N_2^{A-AC} = \left(\left\| \hat{\rho}_2^{T_{A-AC}} \right\|_1 - 1 \right)$, measure the 2-way coherences involving the pairs of subsystems AB and AC , respectively. For the state, $\hat{\rho}^{ABC}$ of Eq. (8) we get,

$$N_2^A = 2\sqrt{\frac{2\mu_0\mu_1}{3}}, \quad N_2^{A-AB} = 2\sqrt{\frac{\mu_0\mu_1}{3}}, \quad (11)$$

and

$$N_2^{A-AC} = 2\sqrt{\frac{\mu_0\mu_1}{3}}. \quad (12)$$

The squared negativities satisfy the monogamy relation

$$(N_2^A)^2 = (N_2^{A-AC})^2 + (N_2^{A-AB})^2. \quad (13)$$

If party C measures the state of qubit three and finds it in state $|0\rangle$ (this event happens with a probability $P_0 = \frac{2\mu_0+1}{3}$) and communicates the result to parties A and B , the entangled state

$$\Phi_0^{AB} = \sqrt{\frac{3\mu_0}{2\mu_0+1}} |00\rangle + \sqrt{\frac{\mu_1}{2\mu_0+1}} |11\rangle \quad (14)$$

becomes available to A and B . In case the third qubit is found to be in state $|1\rangle$ (Probability $P_1 = \frac{2\mu_1}{3}$), the state available to A and B is $\Phi_1^{AB} = (|10\rangle + |11\rangle)/\sqrt{2}$, a separable state. The negativity of partial transpose of the state $|\Phi_0^{AB}\rangle\langle\Phi_0^{AB}|$ is $N^A(|\Phi_0^{AB}\rangle\langle\Phi_0^{AB}|) = 2\sqrt{\frac{3\mu_0\mu_1}{2\mu_0+1}}$, as such the total free entanglement available to A and B is $P_0 N^A(|\Phi_0^{AB}\rangle\langle\Phi_0^{AB}|) = 2\sqrt{\frac{\mu_0\mu_1}{3}}$, which is the same as N_2^{A-AB} obtained from the full state operator. If no communication between parties takes place, the state tomography by A and B should find the qubit one and two to be in a mixed state. The partially transposed matrix obtained from

$$\hat{\rho}^{AB}(\text{red}) = \text{tr}_C(\hat{\rho}^{ABC}) = P_0 |\Phi_0^{AB}\rangle\langle\Phi_0^{AB}| + P_1 |\Phi_1^{AB}\rangle\langle\Phi_1^{AB}|, \quad (15)$$

reads as

$$(\rho^{AB}(\text{red}))^{T_A} = \begin{bmatrix} \mu_0 & 0 & 0 & 0 \\ 0 & \frac{\mu_1}{3} & \sqrt{\frac{\mu_0\mu_1}{3}} & \frac{\mu_1}{3} \\ 0 & \sqrt{\frac{\mu_0\mu_1}{3}} & 0 & 0 \\ 0 & \frac{\mu_1}{3} & 0 & \frac{2\mu_1}{3} \end{bmatrix}. \quad (16)$$

The negativity of partial transpose $(\rho^{AB}(\text{red}))^{T_A}$ satisfies

$$N_G^A((\rho^{AB}(\text{red}))^{T_A}) \leq P_0 N^A(|\Phi_0^{AB}\rangle\langle\Phi_0^{AB}|) + P_1 N^A(|\Phi_1^{AB}\rangle\langle\Phi_1^{AB}|). \quad (17)$$

Local unitary operations on subsystems A and B may transform the states $|\Phi_0^{AB}\rangle$ and $|\Phi_1^{AB}\rangle$ to $|\chi_0^{AB}\rangle$ and $|\chi_1^{AB}\rangle$, respectively with different values of negativities. Although left hand side of Eq. (17) is invariant under such operations, the right hand side may be optimized to obtain a probability distribution that allows the parties A and B to have at their disposal, at least one state with large bipartite entanglement and high probability. We notice that a mixed quantum state is very different from a classical mixed state. No wonder that $N_G^A((\rho^{AB}(\text{red}))^{T_A})$, sometimes, fails to detect the optimum entanglement of subsystem AB which can become available by local operations and classical communication with C . PPT entangled states are a class of states for which the entanglement detected by global negativity turns out to be zero.

It is easily verified that for a tripartite system

$$\hat{\rho}_2^{T_A} = \hat{\rho}_2^{T_{A-AB}} + \hat{\rho}_2^{T_{A-AC}} - \hat{\rho}. \quad (18)$$

Generalization to obtain a measure of K -way coherences involving a specific set of K subsystems from the state operator of N -partite composite system is straight forward. No state reduction is involved here.

III. CONTRIBUTION OF K-WAY NEGATIVITY TO GLOBAL NEGATIVITY

Global negativity with respect to a subsystem p can be written as a sum of partial K -way negativities. Using $\text{Tr}(\hat{\rho}_G^{T_p}) = 1$, the negativity of $\hat{\rho}_G^{T_p}$ is given by

$$N_G^p = -\frac{2}{d_p-1} \sum_i \langle \Psi_i^{G-} | \hat{\rho}_G^{T_p} | \Psi_i^{G-} \rangle = -\frac{2}{d_p-1} \sum_i \lambda_i^{G-}, \quad (19)$$

where λ_i^{G+} and $|\Psi_i^{G+}\rangle$ (λ_i^{G-} and $|\Psi_i^{G-}\rangle$) are, respectively, the positive (negative) eigenvalues and eigenvectors of $\hat{\rho}_G^{T_p}$. The global transpose with respect to subsystem p , may also be written as

$$\hat{\rho}_G^{T_p} = \sum_{K=2}^N \hat{\rho}_K^{T_p} - (N-2)\hat{\rho}. \quad (20)$$

Substituting Eq. (20) in Eq. (19), and recalling that $\hat{\rho}$ is a positive operator with trace one, we get

$$-\frac{2}{d_p-1} \sum_i \lambda_i^{G-} = -\frac{2}{d_p-1} \sum_{K=2}^N \sum_i \langle \Psi_i^{G-} | \hat{\rho}_K^{T_p} | \Psi_i^{G-} \rangle + \frac{2(N-2)}{d_p-1} \sum_i \langle \Psi_i^{G-} | \hat{\rho} | \Psi_i^{G-} \rangle. \quad (21)$$

Defining the partial K -way negativity E_K^p ($K = 2$ to N) as

$$E_K^p = -\frac{2}{d_p-1} \sum_i \langle \Psi_i^{G-} | \hat{\rho}_K^{T_p} | \Psi_i^{G-} \rangle, \quad (22)$$

we may split the global negativity for qubit p as

$$N_G^p = \sum_{K=2}^N E_K^p - E_0^p, \quad (23)$$

where

$$E_0^p = -\frac{2(N-2)}{d_p-1} \sum_{K=2}^N \sum_i \langle \Psi_i^{G-} | \hat{\rho} | \Psi_i^{G-} \rangle. \quad (24)$$

An interesting result is obtained when the global partial transpose has a single negative eigen value that is

$$\lambda^{G-} | \Psi^{G-} \rangle \langle \Psi^{G-} | = \sum_K \sum_m \lambda_m^{K-} | \Psi_m^{K-} \rangle \langle \Psi_m^{K-} |. \quad (25)$$

In this case $\lambda^{G-} = \sum_K \sum_m \lambda_m^{K-}$ leading to

$$(N_G^p)^2 = 4 \sum_{K=2}^N \sum_m (\lambda_m^{K-})^2. \quad (26)$$

For the state $\hat{\rho}^{ABC}$ of Eq. (8), the values

$$N_G^A = 2\sqrt{\mu_0\mu_1}, \quad N_3^A = 2\sqrt{\frac{\mu_0\mu_1}{3}}, \quad N_2^A = 2\sqrt{\frac{2\mu_0\mu_1}{3}}, \quad (27)$$

are obtained when the global, 2-way, and 3-way partial transposes are taken with respect to first qubit (subsystem A). Using the negative eigen values and eigenvectors of $\hat{\rho}_G^{T_A}$ one further gets the measures of GHZ like entanglement $E_3^A = \frac{2}{3}\sqrt{\mu_0\mu_1}$ and the measure of bipartite entanglement $E_2^A = \frac{4}{3}\sqrt{\mu_0\mu_1}$. The necessary condition for an N -partite pure state not to have genuine N -partite entanglement is that at least one of the global negativities is zero that is $N_G^p = 0$, where p is one of the subsystems or one part of a bipartite split of the composite system. Recalling that

$N_G^p = \sum_{K=2}^N E_K^p - E_0^p$ for an N -partite system in a pure state $\hat{\rho}$, the separability of subsystem p implies that $E_K^p \leq 0$,

or $\hat{\rho}_K^{T_p} \geq 0$, for $K = 2$ to N . In general, for a system having only genuine K -partite entanglement, $N_G^p = 0$ for $N - K$ of the subsystems and $E_K^p > 0$ for at least K subsystems. In addition, the lowest positive value of the partial non zero K -way negativities determines the K -partite entanglement, the same being a collective property of K -subsystems.

IV. LOCAL UNITARY OPERATIONS AND K-WAY COHERENCES

An important point to note is that the trace norm $\|\hat{\rho}_K^{T_p}\|_1$ is not invariant under local unitary rotations, unless, $\|\hat{\rho}_K^{T_p}\|_1 = \|\hat{\rho}_G^{T_p}\|_1$. Any composite system state $\hat{\rho}_2$, obtained from an N -partite state $\hat{\rho}_1$ through entanglement

TABLE I: The global, 2-way and 3-way negativities of W-like state $|\Psi_I\rangle_{123}$ and GHZ like state $|\Psi_F\rangle_{123}$. The measures E_2^p and E_3^p ($p = 1 - 3$) are also listed.

	N_G^1	N_2^1	N_3^1	E_2^1	E_3^1
$ \Psi_I\rangle_{123}$	$2\sqrt{a - a^2}$	$2\sqrt{a - a^2}$	0	$2\sqrt{a - a^2}$	0
$ \Psi_F\rangle_{123}$	$2\sqrt{a - 2a^2}$	0	$2\sqrt{a - 2a^2}$	0	$2\sqrt{a - 2a^2}$
	N_G^2	N_2^2	N_3^2	E_2^2	E_3^2
$ \Psi_I\rangle_{123}$	$2\sqrt{a - a^2}$	$2\sqrt{a - a^2}$	0	$2\sqrt{a - a^2}$	0
$ \Psi_F\rangle_{123}$	$2\sqrt{2a(1 - 2a)}$	$2\sqrt{a - 2a^2}$	$2\sqrt{a - 2a^2}$	$\sqrt{2a(1 - 2a)}$	$\sqrt{2a(1 - 2a)}$
	N_G^3	N_2^3	N_3^3	E_2^3	E_3^3
$ \Psi_I\rangle_{123}$	$2\sqrt{2a(1 - 2a)}$	$2\sqrt{2a(1 - 2a)}$	0	$2\sqrt{2a(1 - 2a)}$	0
$ \Psi_F\rangle_{123}$	$2\sqrt{2a(1 - 2a)}$	$2\sqrt{a - 2a^2}$	$2\sqrt{a - 2a^2}$	$\sqrt{2a(1 - 2a)}$	$\sqrt{2a(1 - 2a)}$

conserving local operations, differs from the former in being characterized by a different set of K -way Negativities. A canonical state $\hat{\rho}_c$ obtained from $\hat{\rho}$ through entanglement conserving local operations is a state written in terms of the minimum number of local basis product states [10]. An N-partite system has N-partite entanglement if for all possible splits of the system into two subsystems, the global negativity is non zero. In case N-partite entanglement is generated by N-way coherences, the system has genuine N-partite entanglement equal to $\min(E_N^1, E_N^2, \dots, E_N^N)$, where $E_N^1, E_N^2, \dots, E_N^N$ are calculated from partial transposes constructed from the state $\hat{\rho}_c$. We conjecture that for the state $\hat{\rho}_c$ the quantity E_K^p , as defined in Eq. (22) measures the K -way entanglement of subsystem p with its complement. The motivation for using the canonical state stems from the fact that the coefficients in a canonical form are all local invariants. As such the calculated contributions E_K^p , being functions of local invariants having unique values, qualify to be entanglement measures for all the states lying on the orbit. Consider a three qubit W-like state

$$|\Psi_I\rangle_{123} = \sqrt{a}|100\rangle + \sqrt{a}|010\rangle + \sqrt{1-2a}|001\rangle, \quad a \in \left[\frac{1}{3}, \frac{1}{2}\right] \quad (28)$$

shared by Alice, Bob and Charlie. This state has no 3-way coherences and cannot be converted to a GHZ like state by local operations. If two of the parties, say Alice and Bob get together and perform a CNot gate with qubit one as control and qubit 2 as target qubit, the transformed state

$$|\Psi_F\rangle_{123} = \sqrt{a}|010\rangle + \sqrt{a}|110\rangle + \sqrt{1-2a}|001\rangle, \quad (29)$$

is a GHZ like state. What happens to the coherences during this transformation is manifest in the set of global, 2-way and 3-way negativities associated with the initial and the final state listed in Table I. All the qubits in state $|\Psi_F\rangle_{123}$, have 3-way coherences. In addition qubit two and three also show 2-way coherence. The bipartite entanglement of qubit two and three in the reduced state $\text{tr}_1(|\Psi_F\rangle_{123}\langle\Psi_F|)$ is due to coherences $N_2^2 = N_3^2 = 2\sqrt{a - 2a^2}$. The state reduction destroys genuine tripartite entanglement and increase the bipartite entanglement.

V. WHAT DOES THE N-WAY NEGATIVITY TELL ABOUT THE ENTANGLEMENT AVAILABLE TO N PARTIES SHARING THE COMPOSITE SYSTEM?

Let $\nu_N(\rho)$ be the number of negative eigen values of $\rho_N^{T_p^-}$ of an entangled state

$$\rho = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle = \sum_i |\Phi_i\rangle, \quad (30)$$

where

$$\begin{aligned} |\Phi_i\rangle &= a_i |i_1 i_2 \dots i_N\rangle + b_i |j_1 j_2 \dots j_N\rangle, \quad i_m \neq j_m, (m = 1 \text{ to } N), \\ \langle\Phi_j|\Phi_i\rangle &= \sqrt{|a_i|^2 + |b_i|^2} \delta_{ij}, \end{aligned} \quad (31)$$

with qubits 1, 2, 3, ... held by parties A, B, C... , respectively. A state $|\Psi\rangle$ with non increasing $\nu_N(\rho)$ under local unitaries indicates that the state may be transformed to a state having minimum number of LBPS by LU on qubit p

alone, and has the same negativity as the canonical state. An N -qubit canonical state may, in turn, be written as

$$|\Psi\rangle_c = \sum_{i=1}^{\nu_N(\rho_c)} |\Phi_i\rangle + X, \quad \rho_c = |\Psi\rangle_{cc} \langle\Psi|, \quad \langle X | \Phi_i \rangle = 0 \text{ for } i = 1 - \nu_N(\rho_c), \quad (32)$$

such that $\nu_N(\rho_c)$, the number of eigen values of $(\rho_c)_N^{T_p-}$, is non increasing under local unitary transformations.

In case the negative part of the N -way partial transpose

$$\rho_N^{T_p-}(|\Psi\rangle \langle\Psi|) = - \sum_{i=1}^{\nu} |a_i| |b_i| |\Psi_i^{N-}\rangle \langle\Psi_i^{N-}|, \quad (33)$$

lies in a state space orthogonal to that of state ρ , we get $N_N^p = 2 \sum_i |a_i| |b_i|$ and

$$|\Psi_i^{N-}\rangle = \left(\frac{|i_1 i_2 \dots i_{p-1} j_p i_{p+1} \dots i_N\rangle - |j_1 j_2 \dots j_{p-1} i_p j_{p+1} \dots j_N\rangle}{\sqrt{2}} \right). \quad (34)$$

For a given state the negativity of N -way partial transpose of $|\Psi\rangle$ with respect to a qubit is the sum of negativities of ν_N unnormalized GHZ like states $|\Phi_i\rangle$ that may be projected out of the state. In general, the K -way negativity of partial transpose with respect to qubit p depends on the negativities of ν_K unnormalized GHZ like states of K qubits (where p is one of the K qubits) that may be projected out of the state. The information content of a multipartite state is distributed over $\nu = (\sum_{K=2-N} \nu_K)$ states, with a specific pobability distribution. The value of ν is minimized by unitary operations leading to the canonical state, which is a state written in terms of the lowest number of LBPS. A canonical state optimizes the probability distribution of projecting out GHZ-like states. In a canonical state, the total information content of the composite state, is shared by the lowest number of GHZ like states that may be projected out of state. The motivation for considering partial K -way negativities of the canonical state as entanglement measures resides in the fact that a canonical state optimizes the form in which the entanglement is available to a given set of K -parties, for all possible sets of K -parties and all possible values of K .

VI. SCHMIDT DECOMPOSITION AND K -WAY NEGATIVITIES

Consider a set of four parameter two qubit (qubits A and B) and one qutrit (C) states

$$\Psi = a_0 |000\rangle + a_1 |101\rangle + a_2 |011\rangle + a_3 |112\rangle, \quad \sum |a_i|^2 = 1. \quad (35)$$

The state can be easily written in Schmidt form with respect to qubit A or qubit B, with Schmidt coefficients,

$$\mu_{0A} = \sqrt{|a_0|^2 + |a_2|^2}, \quad \mu_{1A} = \sqrt{|a_1|^2 + |a_3|^2}, \quad (36)$$

and

$$\mu_{0B} = \sqrt{|a_0|^2 + |a_1|^2}, \quad \mu_{1B} = \sqrt{|a_2|^2 + |a_3|^2}. \quad (37)$$

The calculated negativities

$$\begin{aligned} N_G^A &= N_G^B = 2\mu_{0A}\mu_{1A}, \\ N_3^A &= N_3^B = 2|a_0||a_3|, \quad N_2^A = 2|a_0||a_1| + 2|a_2|\mu_{1A}, \end{aligned} \quad (38)$$

and

$$N_2^B = 2|a_0||a_2| + 2|a_1|\mu_{1B}, \quad (39)$$

satisfy the relation

$$(N_G^A)^2 \leq ((N_2^A)^2 + (N_3^A)^2), \quad (N_G^B)^2 \leq ((N_2^B)^2 + (N_3^B)^2), \quad (40)$$

and the entanglement measures for the state are

$$E_2^A = 2 \frac{|a_0|^2 |a_1|^2 + |a_2|^2 \mu_{1A}^2}{\mu_{0A} \mu_{1A}}, \quad E_3^A = E_3^B = 2 \frac{|a_0|^2 |a_3|^2}{\mu_{0A} \mu_{1A}}, \quad E_2^B = 2 \frac{|a_0|^2 |a_2|^2 + |a_1|^2 \mu_{1B}^2}{\mu_{0B} \mu_{1B}}. \quad (41)$$

Rewriting $\hat{\rho}_2^{T^A}$ and $\hat{\rho}_2^{T^B}$ as

$$\hat{\rho}_2^{T^A} = \hat{\rho}_2^{T^{A-AB}} + \hat{\rho}_2^{T^{A-AC}} - \hat{\rho}, \quad \hat{\rho}_2^{T^B} = \hat{\rho}_2^{T^{B-AB}} + \hat{\rho}_2^{T^{B-BC}} - \hat{\rho}, \quad (42)$$

we identify

$$N_2^{A-AB} = 2 |a_1| |a_2|, \quad N_2^{A-AC} = 2 |a_0| |a_1| + 2 |a_2| |a_3|, \quad (43)$$

and

$$N_2^{B-BC} = 2 |a_1| |a_3| + 2 |a_0| |a_2|. \quad (44)$$

The bipartite entanglement measures defined as in Eq. (23) have the structure $E_2^A = E_2^{AB} + E_2^{AC}$ and $E_2^B = E_2^{AB} + E_2^{BC}$, where

$$E_2^{AB} = 2 \frac{|a_2|^2 |a_1|^2}{\mu_{0A} \mu_{1A}}, \quad E_2^{AC} = 2 \frac{|a_0|^2 |a_1|^2 + |a_2|^2 |a_3|^2}{\mu_{0A} \mu_{1A}}, \quad (45)$$

$$E_2^{BC} = 2 \frac{|a_0|^2 |a_2|^2 + |a_1|^2 |a_3|^2}{\mu_{0B} \mu_{1B}}. \quad (46)$$

The number of local basis product states in state of Eq. (35) can not be reduced by local rotations. The partial K-way negativities E_2^{AB} , E_2^{AC} and E_2^{BC} calculated from pure state operator determine the bipartite entanglement. The entanglement available to A and B if they have no knowledge of C , as well as the probabilistic entanglement of subsystem AB after a measurement has been made by C , is determined by E_2^{AB} . Whenever, a multipartite state can be written in Schmidt form for some of the subsystems, analytical expressions for entanglement measures are easily found. When no analytical expressions are available, numerical calculations using standard subroutines for calculating eigenvalues come handy.

To summarize, we have defined global and K -way negativities calculated from global and K -way partial transposes, respectively, of an N -partite state operator. For a given partition of a multipartite quantum system, global negativity measures overall entanglement of parties. The K -way negativities for $2 \leq K \leq N$, on the other hand, provide a measure of K -way coherences of the system. Entanglement is invariant with respect to local unitary operations, whereas, coherences are not so. The K -way negativity of partial transpose with respect to a subsystem depends on the negativities of unnormalized K subsystem GHZ like states available to K parties. A canonical state is special in that the information contained in the state is shared by the minimum possible number of K- subsystem GHZ like states ($2 \leq K \leq N$). Global negativity with respect to a subsystem can be written as a sum of partial K -way negativities. We conjecture that the partial K -way negativities provide measures of K -way entanglement for N -partite canonical states. Analysis of K-way negativities is expected to provide insight into the entanglement distribution amongst different parts of a quantum system and point out the direction of entanglement flow during processes involving time evolution of composite quantum system under unitary operations. We believe that the use of K-way negativities to characterize entangled states is an important step towards the understanding of quantum correlations. As local operations transform K -partite entanglement to entanglement available to K' parties, the use of K -way negativities should be helpful in finding the multipartite state that optimizes the entanglement distribution for implementation of a specific quantum computation and communication related task. We recall that the multipartite entanglement is still not well understood, even for low-dimensional quantum systems. The ideas presented in this article, just point out a different direction in which one can look for solutions.

Financial support from CNPq, Brazil and Faep-Uel, Brazil is acknowledged.

-
- [1] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
 - [2] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 8 (1996).
 - [3] P. Horodecki, Phys. Lett. A 232, 333 (1997).
 - [4] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).

- [5] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A 58, 883 (1998).
- [6] G. Vidal and R. F. Werner, Phys. Rev. Vol. 65, 032314 (2002).
- [7] J. Eisert, PhD thesis (University of Potsdam, February 2001).
- [8] S. Shelly Sharma and N. K. Sharma, Phys. Rev. A 76, 012326 (2007).
- [9] S. Shelly Sharma and N. K. Sharma, arXiv: quant-ph/0608062 (unpublished).
- [10] H. A. Carteret, N. Linden, S. Popescu, and A. Sudbery, Foundations of Physics, Vol. 29, No. 4, 527 (1999); H. A. Carteret, A. Higuchi, A. Sudbery, J. Math. Phys. 41, 7932 (2000).
- [11] A. Acín, A. Andrianov, L. Costa, E. Jane, J.I. Latorre, R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000); A Acín, A Andrianov, E Jané and R Tarrach, J. Phys. A: Math. Gen. 34 6725-6739 (2001).